A Non-Standard Proof of Clairaut’s Theorem for the Symmetry of Partial Derivatives

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Abstract

We prove Clairaut’s theorem (also known as Schwarz’ theorem) in a somewhat unconventional manner. More typical proofs might start from the definition of the partial derivative in terms of limits \[1\] \[2\]. Other relatively common proofs exist \[3\], but some are more unique \[3\] \[3\].

The author “discovered”, for lack of a better word, this proof when answering a question on the Q & A site Quora. While it appears the author was not the first to think of this proof, as it is mentioned and hinted at on various websites, it is difficult if not impossible to find an example of the proof in full online.

Clairaut’s theorem: If a function \( f : \mathbb{R}^2 \to \mathbb{R} \) has globally continuous second partial derivatives, then \( f_{xy}(a, b) = f_{yx}(a, b) \) for any point \( P = (a, b) \in \mathbb{R}^2 \).

Proof: Consider the gradient of some function, \( f, \nabla f(x, y) : \mathbb{R}^2 \to \mathbb{R}^2 \) where \( \nabla f(x, y) = (f_x, f_y) \). The line integral of the gradient around some square region \( C \) (which is positively oriented, piece-wise smooth, simple, and closed – this will be important later) must be zero.

\[
\int_C f_x \, dx + f_y \, dy = 0
\] (1)

This is due to the fundamental theorem of line integrals.

**Lemma 1 (The Fundamental Theorem of Line Integrals):**

\[
\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))
\]

where \( a \) is the initial point on the path and \( b \) is the final point on the path \( C \).

**Proof:** This is left to the reader.

\[\blacksquare\]

We will need a second lemma to prove this theorem, namely Green’s theorem.

**Lemma 2 (Green’s Theorem):**

For any positively oriented, piece-wise smooth, simple, closed curve \( C \) and for some open region \( D \) enclosed by \( C \) we have that if \( P \) and \( Q \) have continuous partial derivatives on \( D \), then

\[
\int_C P \, dx + Q \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA
\]

**Proof:** This is left to the reader.

\[\blacksquare\]
Hence, by Green’s theorem (1) is equivalent to

\[ \int \int \limits_{D} (f_{yx} - f_{xy}) \, dx \, dy \]  

(2)

It is important to note that the gradient and our square region satisfy all the hypotheses of Green’s theorem.

In other words, this says that (1) is equivalent to (2) is equivalent to 0.

But, if \( f_{yx} - f_{xy} \neq 0 \) for some point \( P = (a, b) \), i.e. if \( f_{xy} \neq f_{yx} \) for some point, then by continuity we have that there exists some square around \( P \) such that the integral is non-zero, thus contradicting Green’s theorem.

So, we have arrived the desired result via proof by contradiction. Specifically, we must have that \( f_{xy} = f_{yx} \) when the second partials of \( f \) are defined and continuous in some open region.

\[ \blacksquare \]

One can easily extend this to prove Clairaut’s theorem in its most general form.

**Clairaut’s theorem (General Form):** If \( f : \mathbb{R}^n \to \mathbb{R} \), whose domain elements are \( x_1, x_2, \ldots, x_i \), has continuous second partial derivatives at any given point \( P = (a_1, \ldots, a_n) \in \mathbb{R}^n \), then for all \( i, j \in \{1, 2, \ldots, n - 1, n\} \)

\[ \frac{\partial^2 f}{\partial x_i \partial x_j} (a_1, \ldots, a_n) = \frac{\partial^2 f}{\partial x_j \partial x_i} (a_1, \ldots, a_n) \]

**Proof:** This is left to the reader.

\[ \blacksquare \]

**References**

[1] [http://homepage.smc.edu/nestler_andrew/math11/m11clairaut.pdf](http://homepage.smc.edu/nestler_andrew/math11/m11clairaut.pdf)


[3] [http://www.math.ubc.ca/~feldman/m105/mixedPartials.pdf](http://www.math.ubc.ca/~feldman/m105/mixedPartials.pdf)

[4] [http://people.sju.edu/~pklingsb/clairaut](http://people.sju.edu/~pklingsb/clairaut)